Notes for Differential Topology (Math 214)

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September 2025

1 Notes

Let $f_0, f_1 : M \to N$ be immersions. A **regular homotopy** is a smooth map

$$F : [0,1] \times M \to N$$

such that $F(0,\cdot) = f_0, F(1,\cdot) = f_1$ and for all $t \in [0,1] : f_t : M \to N$ the map sending $x \mapsto F(t,x)$ is an immersion.

Remark Classify immersions $S^1 \to \mathbb{R}^2$/regular homotopy. For $\gamma : S^1 \to S^1$, we define the degree to be the winding number. This agrees with the standard definition on homology.

Whitney Graustein Theorem Two immersions $S^1 \to \mathbb{R}^2$ regular homotopic iff they have the same winding number.

Immersions are locally immersions Lee proves this for constant rank maps. We do it for rank $df_p = \dim M$ for $\dim M \leq \dim N$, $F : M \to N$. Let Then after applying elementary matrix transformation (we count them as part of the chart) we can make the matrix of df_p look like

$$\begin{pmatrix} I_{m\times m} \\ 0 \end{pmatrix}_{n\times m}$$

Define $g : \mathbb{R}^m \to \mathbb{R}^{n-m} \to \mathbb{R}^n$, $g(x,y) = f(x) + y$. Then

$$dg_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

By inverse function theorem we can find $h : 0 \in U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ g = id$. So $h(f(x)) = (x, 0)$. We can just compose h into the coordinate charts (trestricting the domain as necessary).

Define an **embedding** to be an injective immersion $f : M \to N$ such that $f(M)$ is a submanifold of N and the map $M \to f(M)$ is a diffeomorphism.

A **submanifold** is a subset $S \subset M$ such that it is locally given as inclusion. Formally, for every $p \in S$ there exists (U,φ) such that $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

I was confused on how this is consistent with the definitions given in Lee: (Lee) An **embdding** is a smooth immersion that is also a topological embedding (e.g. $M \to F(M)$ is a homeomorphism). (Lee) A **submanifold** is a subset $S \subset M$ such that $\iota : S \to M$ is a smooth embedding.

Their equivalence can be proven using Theorem 4.25 in Lee:
For $F : M \to N$, it is a smooth immersion iff it is locally a (Lee) smooth embedding.
On then shows for the definition given in class, S does inherit the subspace topology of M (e.g. $S \cong \iota(S) \subset M$, e.g. a topological embedding).

Theorem. If $f : M \to N$ is an injective immersion and M is compact, then f is an embedding. Follows from closed map theorem.

Example where this fails: irrational flow on the torus. Let $a, b \in \mathbb{R}$ such that $\frac{a}{b} \notin \mathbb{Q}$. Define $f : \mathbb{R} \to \mathbb{T}^2$ by $f(t) = (at \mod 1, bt \mod 1)$. It is an immersion because $f'(t) = (a, b) \neq (0, 0)$. It is injective because

$$f(t_1) = f(t_2) \Rightarrow a(t_2 - t_1) = m, b(t_2 - t_1) = n \Rightarrow \frac{a}{b} = \frac{m}{n} \in \mathbb{Q}$$

The image $f(\mathbb{R})$ is dense in $\mathbb{T}^2$, so we can not find neighborhood such that $\mathbb{R}$ "looks" like a local line.

For $F : M \to N, q \in N$ is a **regular value** if for all $p \in F^{-1}(q)$, dF_p is surjective.

Theorem (Implicit function theorem for manifolds)

If q is a regular value of f, $f^{-1}(q) \neq \emptyset$, then $f^{-1}(q)$ is a submanifold of M. Moreover, if $p \in f^{-1}(q)$ then $T_p f^{-1}(q) = \ker df_p$. $\dim f^{-1}(q) = \dim M - \dim N$. Prove this next lecture.

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Example: $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y) = x^2 + y^2 - z^2$. Then

$$df = 2xdx + 2ydy + 2zdz = 0 \Rightarrow (x, y, z) = (0, 0, 0), f = 0$$

So every nonzero real is a regular value of f.

Theorem. 1

Let $f : \mathbb{R}^m \to \mathbb{R}^n$, with $m > n$. Let $x = (x^1, \dots, x^{m-n}), y = (y^1, \dots, y^n), (x, y) \in \mathbb{R}^n$ and $z = (z^1, \dots, z^m) \in \mathbb{R}^m$. Suppose $f(0) = 0$, and $\left(\frac{\partial f^i}{\partial y^j}(0, y)\right)$ is invertible. Then there exists smooth function $g : 0 \in U \subset \mathbb{R}^{m-n} \to \mathbb{R}^n$ such that $f^{-1}(0) \cap V \subset \mathbb{R}^m = \Gamma(g)$.

Example. 1

$f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 + y^3 + z + z^5$. Then $\frac{\partial f}{\partial z}(0, 0, z) = 1 + 5z^4 > 0$. Around zero, $f^{-1}(0)$ can be represented as a graph of $0 \in U \in \mathbb{R}^2$.

Proof. Define $h : \mathbb{R}^m \to \mathbb{R}^m$ by $h(x, y) = (x, f(x, y))$. Then the matrix of dh_0 looks like

$$\begin{pmatrix} I_{m-n} & 0 \\ * & A \end{pmatrix}$$

where A is invertible by hypothesis. Hence determinant nonzero. By IFT h^{-1} is defined and smooth near 0. Define $g(x) = h^{-1}(x, 0)$. This is a point with $f(x, y) = 0$. $(x \in \mathbb{R}^{m-n}, y \in \mathbb{R}^n)$ ∎

Theorem. 2

If M is a compact manifold of dimension n, tjhen there exists an embedding $M \to \mathbb{R}^k$ for some k .We can do it for $k = 2n$.

Example. 2

nonorientable compact 2-manifolds don't embed to $\mathbb{R}^3$, but in $\mathbb{R}^4$.

Proof. Cover M with coordinate charts. Say open $U \subset M$ is nice if there is a coord chart $\varphi : V \to \hat{V} \subset \mathbb{R}^n$ and a closed ball $B \subset \hat{V}$ such that $U = \varphi^{-1}(Int(B))$. The nice open sets cover M. Choose a finite subcover $U_i = \varphi_i^{-1}(B_i)$, $\varphi_i : V_i \to \hat{V}_i$, $i = 1, \dots, l$.
For each i, choose a smooth function $\rho_i : M \to \mathbb{R}$ such that $\rho_i = 1$ on U_i. and $\rho_i = 0$ on $M \backslash V_i$. Define $f : M \to \mathbb{R}^{ln}$ by

$$f(p) = (\rho_1 \varphi_1, \dots, \rho_l \varphi_l)$$

Show f is an embedding. It is injective: Let $p \in M$. Then $p \in U_i$ for some i, then $f(p)$ determines $\rho_i(p)\varphi_i(p) = \varphi_i(p)$ which determines p.
Immersion: Let $p \in M$, choose i with $p \in U_i$, then the i^{th} block of f in a neighborhood of p is given by φ_i. $d(\varphi_i)_p$ is an isomorphism, because φ_i is a diffeomorphism. Can we make k smaller? Let $f : M \to \mathbb{R}^k$ now be an embedding. Let $v \in S^{k-1}$.
Idea: try composing f with the orthogonal projection $\mathbb{R}^k \to V^\perp \cong \mathbb{R}^{k-1}$. Claim: If $k > 2n + 1$, then there exists v for which this composition is an embedding. When is projection of M to $V^\perp$ an embedding? To start, when is it injective?

Define $g_i(M \times M)\backslash\Delta \to S^{k-1}$. The diagonal $\Delta = \{(p,p) : p \in M\}$. By $g(p,q) = \frac{p-q}{|p-q|}$. The projection $M \to V^{\perp}$ is injective iff $v \notin \text{im } g$. Want: Image of g is not all of S^{k-1}. Intuition: image of g has "dimension $\leq 2n$". If $2n < k-1$ then g is not surjective. $\blacksquare$

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Theorem. 3
Sard's theorem. The set of critical values of f has measure zero.

Example. 3
A submanifold of codimension ¿ 0 has measure 0

Proof. We do an easy case $M = S^1 = \mathbb{R}/\mathbb{Z}, N = \mathbb{R}$. Can regard $df : S^1 \to \mathbb{R}$. Given $\epsilon > 0$, let $X_\epsilon = \{p \in S^1 || df_p| < \epsilon\}$. Observe that $f(X_\epsilon)$ has measure $\leq \epsilon$. The set of critical values of f is $\bigcap_{\epsilon>0} f(X_\epsilon)$. This has measure zero. ∎

Example. 4
The set of regular values is dense in N.

We continue to prove Whitney's embedding theorem

Proof. $M \subset \mathbb{R}^k$ compact submanifold, $\dim(M) = n$, assume $k \geq 2n + 2$. Given $v \in S^{k-1}$, define $\pi_V : \mathbb{R}^k \to V^\perp \cong \mathbb{R}^{k-1}$ by the orthogonal projection.

Claim: there exists $v \in S^{k-1}$ such that $\pi_V|_M : M \to V^\perp$ is an embedding. Need $\pi_V|_M$ is injective and an immersion. Define $f(M \times M)\backslash\Delta \to S^{k-1}$ by $f(p, q) = \frac{p-q}{\|p-q\|}$. Observe that given $v \in S^{k-1}$ we have

$$\pi_V(p) = \pi_V(q) \Leftrightarrow f(p, q) = \pm v$$

So $\pi_V|M$ being injective is equivalent to $v, -v \notin f$. By Sard's, the set of critical values of f has measure zero in S^{k-1}. So there exist $v \in S^{k-1}$ such that neither v nor $-v$ is a critical value of f. since $\dim((M \times M)\backslash\Delta) = 2n < k - 1$ (so $\pm v$ can not be regular points) that implies $\pm v$ are not in the image of f.
To complete the proof, we need to show that for v in the complement of the measure zero set in S^{k-1} the proejction $\pi_V|_M : M \to V^\perp$ is an immersion. Observe that $d(\pi_V|_M)_p$ fails to be injective $\Leftrightarrow v \in T_pM$. Define the unit tangent bundle of M by $STM = \{(p, w)| p \in M, w \in T_pM \subset \mathbb{R}^n, \|w\| = 1\}$. The STM is a smooth manifold. Smooth map $g : STM \to S^{k-1}$ by $(p, w) \mapsto w$. Since $\dim(STM) = 2n - 1 < k - 1 = \dim(S^{k-1})$ by Sard's theorem the image of g has measure zero. If $v \notin$ image g then $\pi_V|_M$ is an immersion. ∎

Transversality
Question: Let M be a smooth manifold. Let X, Y be submanifolds of M. When is $X \cap Y$ a submanifold?
Definition. X, Y intersect transversely $X \pitchfork Y$ if $\forall p \in X \cap Y, T_pX + T_pY = T_pM$.

If $X \pitchfork Y$ then $X \cap Y$ is a submanifold of M and $\dim(X \cap Y) = \dim(X) + \dim(Y) - \dim(M)$ and for each $p \in X \cap Y$,

$$T_p(X \cap Y) = T_pX \cap T_pY$$

(T_pM is in this intersection).
A more general version: For smooth $f : M \to N \supseteq Y$. f is transverse to Y if for all $p \in f^{-1}(Y)$

$$df_p(T_pM) + T_{f(p)}Y = T_{f(p)}N$$

$f = incl$ recovers previous definition.

If Y is a point in N, then f is transverse to Y iff Y is a regular value of f. If f is transverse to Y, then $f^{-1}(Y)$ is a submanifold of M with dimension

$$\dim(f^{-1}(Y)) = \dim(M) + \dim(Y) - \dim(N)$$

If $p \in f^{-1}(Y)$ then $T_p f^{-1}(Y) = df_p^{-1}(T_{f(p)}Y)$.

Proof. WLOG $N = \mathbb{R}^n, Y = \{(y^1, \dots, y^k, 0, \dots, 0)\} = \mathbb{R}^k \subset \mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k} = \{(0, \dots, 0, x^1, \dots, x^{n-k})\}$ by the projection. Then

$$f^{-1}(Y) = (\pi \circ f)^{-1}(0)$$

Since f is transverse to Y, if $p \in f^{-1}(Y)$ then $df_p(T_pM) + T_{f(p)}Y = \mathbb{R}^n$. This implies

$$d(\pi \circ f)_p(T_pM) = \pi(df_p(T_pM)) = \mathbb{R}^{n-k}$$

Hence 0 is a regular value of $\pi \circ f$, so by implicit function theorem $(\pi \circ f)^{-1}(0)$ is a submanifold of M.

Theorem. 4

Informally: Given $f : M \to N$ and $Y \subset N$, we can perturb f to make it transverse.

More precisely: Suppose N is a compact submanifold of $\mathbb{R}^k$.

Fact: There is a neighborhood $U \subset N \subset \mathbb{R}^k$ and a well defined map $\pi : U \to N$. Sending $x \in U$ to the unique point $y \in N$ closest to x, Given small $v \in \mathbb{R}^k$, define $f_v : M \to N$ by

$$f_v(p) = \pi(f(p) + v)$$

Theorem: f_v is transverse to Y for v in the complement of a measure zero set.

■

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Let M be a smooth manifold, X, Y compact submanifolds of M. If $\dim X + \dim Y = \dim M$ then $\dim X \cap Y = 0$ and is compact. i.e. a finite set of points.

Orientations Let V be a finite dimensional real vector space, an orientation of V is the equivalence class of bases for V defined as follows:
for two bases $(x_1, \dots, x_n), (y_1, \dots, y_n)$ let $x_i = A_i^j y_j$ i.e. A is the change of basis matrix. They are equivalent if $\det A > 0$. There are two orientations because you can compose the change of basis matrices.

Default orientation of $\mathbb{R}^n$ is given by $(e_1, \dots, e_n)$ where $e_1, \dots, e_n$ are the standard basis vectors. Let M be a smooth n-manifold. An orientation of M is an orientation of the tangent space T_pM for all $p \in M$, and depends continuously on p. Continuity means that if $\varphi : U \to \hat{U}$ is a coordinate chart, then either for each $p \in U$ the orientation of T_pM is given by $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ or its opposite. i.e. it remains the same in every coordinate chart.
Definition: M is orientable if there exists an orientation of M. In this case, the number of orientations is 2^k where k is the number of components of M.

Example. 5

S^2 is orientable. $T_pS^2 = p^\perp \subset \mathbb{R}^3$. For $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z) = x^2 + y^2 + z^2$ get $df = 2xdx + 2ydy + 2zdz$. So $T_pS^2 = \ker df_p = \{y = y^i \partial/\partial x^i \mid df_p(y) = 0\}$. Let (v, w) be an oriented basis for T_pS^2 iff (p, v, w) is an oriented basis for $\mathbb{R}^3$. Can check that this is continuous.

More generally, any compact codim 1 submanifold M of $\mathbb{R}^n$ is orientable. We choose a normal vector to T_pM for each $p \in <$ which depends continuously on p. The normal vector is guarenteed by a generalization to the Jordan Curve theorem on all codim 1 manifolds.

Example. 6

Mobius strip: $\mathbb{R} \times (-1, 1)/\sim$, $(x + 1, y) \sim (x, -y)$.

Suppose M has an orientation. WLOG $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is an oriented basis for $T_{(y_2, 0)}M$. We argue that if we continuously "loop" the normal vector around we get back to the opposite orientation. Coordinate chart $\varphi_1 : U_1 \to \{[(x, y)] \in M \mid x \neq 0, 1\}$ by $[(x, y]) \mapsto (x, y)$. By continuity, $(\partial/\partial x, \partial/\partial y)$ is an oriented basis for T_pM whenever $p = (x, y)$.
Another coordinate chart $\varphi_2 : U_2 \to \{[(u, v)] \in M \mid 1/2 < u < 3/2\}$ by $[(u, v)] \mapsto (u, v)$. Then $U_1 \cap U_2$ has two components.

$$U_1 \cap U_2 = \{[(x, y)] \mid 0 < x < 1/2\} \cup \{[(x, y)] \mid 1/2 < x < 1\}$$

The first set sends $(x, y) \mapsto (x + 1, -y)$ and the second sends $(x, y) \mapsto (x, y)$.
Thus

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}, \frac{\partial}{\partial v} = -\frac{\partial}{\partial y}$$

using our usual change of coordinates rule. The determinant is negative, so $(\partial/\partial u, \partial/\partial v)$ and $(\partial/\partial x, \partial/\partial y)$ cannot both be oriented bases for T_pM. Now let's move on to the second set.
Since $(u, v) = (x, y)$, the oriented basis agree so $(\partial/\partial u, \partial/\partial v)$ is an oriented basis for all of U_2. This gives our desired contradiction.

Example. 7

If M is orientable the tangent bundle is orientable.

Proof. Let $\varphi : U \to \hat{U}$ be a chart of M. The charts are given by $\hat{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$(x^1, \dots, x^n, v^i \partial/\partial x^i) \mapsto (x^1, \dots, x^n, v^1, \dots, v^n)$$

For $p \in U$ and $v \in T_pU$ declare

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}$$

to be an oriented basis for $T_{(p,v)}(TM)$. Need to check that this does not depend on the coord chart. Let $\psi : U' \to \hat{U}'$ be another coord chart with coords $(y^1, \dots, y^n)$, we get coord chart on TM $\hat{\psi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$, sending $(y^1, \dots, y^n, w^i \partial/\partial w^i) \mapsto (y^1, \dots, y^n, w^1, \dots, w^n)$. Need to check: $\pi^{-1}(u) \cap \pi^{-1}(U') = \pi^{-1}(U \cap U')$, that $(\partial x^i, \partial v^i)$ and $(\partial y^i, \partial w^i)$ are equivalent bases. i.e. change of coordinate martrix has positive determinant. Note

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

and

$$A = \begin{pmatrix} \left(\frac{\partial x^j}{\partial y^i}\right) & * \\ 0 & \left(\frac{\partial x^j}{\partial y^i}\right) \end{pmatrix}$$

because

$$\begin{aligned} w &= w^i \partial y^i \\ &= w^i \frac{\partial x^j}{\partial y^i} \partial x^j = v^j \partial x^j \end{aligned}$$

where $v^j = w^i \frac{\partial x^j}{\partial y^i}$. ■

Return to transversality: Let M be an oriented smooth manifold, X, Y be compact submanifolds. Suppose $X \pitchfork Y$ and $\dim X + \dim Y = \dim M$ so $X \cap Y$ is a finite set of points. Assign a sign to each point in $X \cap Y$ as follows:
Let $p \in X \cap Y$. By transversality

$$T_pM = T_pX \oplus T_pY$$

Let $(v_1, \dots, v_k)$ be an oriented basis for T_pX and $(w_1, \dots, w_l)$ be an oriented basis for T_pY. We say that p is a **positive** intersection if $(v_1, \dots, v_k, w_1, \dots, w_l)$ is an oriented basis for T_pM. Otherwise p is a **negative** intersection.
Define the intersection number

$$\#(X \cap Y)$$

to be the number of positive intersections - the number of negative intersections.

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Let $X, Y \subset M$ be compact oriented submanifolds and

$$\dim X + \dim Y = \dim M$$

Define

$$\#(X \cap Y) = \sum_{p \in X \cap Y} \begin{cases} +1 & \text{if } T_pX \oplus T_pY \cong T_pM \\ -1 & \text{otherwise} \end{cases}$$

Technical facts:

- For any compact submanifolds X, Y can do is isotopy of X, Y to obtain ttansversality

- If X is isotopic to X' and Y is isotopic to Y' and $X \pitchfork X'$, $Y \pitchfork Y'$, then $\#(X \cap X') = \#(Y \cap Y')$. For any compact submanifolds X, Y of oriented M with $\dim X + \dim Y = \dim M$, intersection number $= \#(X \cap Y) \in \mathbb{Z}$ is well defined and isotopy invariant.

Vector fields Let M be a smooth manifold. Let $\pi : TM \to M$ denote the tangent bundle. a vector field on M is a smooth map $V : M \to TM$ such that $\pi \circ V = id_M$. That is, V associates to each $p \in M$ a tangent vector $V(p) \in T_pM$ and this depends smoothly on p. In local coordinates $x^1, \ldots, x^n$ can write $V = V^i \partial_{x^i}$ where $V^1, \ldots, V^n$ are smooth functions of $x^1, \ldots, x^n$. A zero fo V is a point $p \in M$ such that $V(p) = 0$.
Question: What can we say about the zeros of V?

Let $Z \in TM$ denote the zero section

$$Z = \{(p, 0) \in TM\} \cong M$$

A zero of V is a point in $V^{-1}(Z)$.
Number of zeros of V should be an intersecton number of Z with $\Gamma(V) = \{(p, (V(p)) \,|\, p \in M\} \subset TM$.
Assume M is oriented (not actually necessary) implies TM is canonically oriented.

When is V transverse to Z?
Consider a point $p \in M$ with $V(p) = 0$, $i_p : T_pM \to TM$ by $v \mapsto (p, v)$ and $\pi : TM \to M$ by $(p, v) \mapsto p$. We have a split short exact sequence

$$0 \to T_pM \xrightarrow{(di_p)_0} TM \xrightarrow{d\pi_{(p,0)}} T_pM \to 0$$

Section $j : T_pM \to TM$ by $p \mapsto (p, 0)$, which implies a canonical isomorphism $T_{(p,0)}TM \cong T_pM \oplus T_pM$.
Rrmark If $(p, v) \in T_pM$ with $v \neq 0$ we still have short exact sequence

$$0 \to T_pM \xrightarrow{(di_p)_v} TM \xrightarrow{d\pi_{(p,v)}} T_pM \to 0$$

but it does not have a canonical splitting. A splitting satisfying certain conditions is called a connection.

Let V be a vector field. A zero of p of V is nondegenerate if V is transverse to Z at $(p, 0)$. In this case, there is a sign associated to p. In coordinates $x^1, \ldots, x^n$ with p corresponding to 0, write $V = V^i \partial_{x^i}$. Then $\text{sign}(0) = \text{sign}(\det(\frac{\partial V^i}{\partial x^j}))$.
$dV_p : T_pM \to T_{(p,0)}(TM)$ Transversality means that $\text{im}(dV_p)$ is complementary to $T_{(p,0)}Z$. Equivalently, the composition

$$T_pM \xrightarrow{dV_p} T_{(p,0)}(TM) \xrightarrow{\Xi} T_pM$$

is an isomorphism. In local coords, $\Xi \circ dV_p = (\frac{\partial V^i}{\partial x^j})$.

If M is a comapct oriented manifold and V is a vector field with nondegenerate zeroes, define $\#(V^{-1}(0)) = \#(Z \cap \Gamma(V)) = \sum_{V(p)=0} \text{sign}(\det(\nabla V_p)) \in \mathbb{Z}$. Technical fact: All vector field M can be perturbed to make all zeroes nondegenerate.

Theorem. 5

(Poincare Hopf index theorem) Let M be a compact oriented smooth manifold, and V is a vector field on M

$$\#V^{-1}(0) = \chi(M)$$

where χ denotes the euler characteristic. Technical fact: M has a triangulation.

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim(H^i_{dR}(M))$$

independent of triangulation. Remark: if M is a compact odd dimensional manifold then $\chi(M) = 0$.

Theorem. 6

(Hairy Balls) If all zeros of V is nondegenerate then $\#(V^{-1}(0)) = 2$. This contradicts poincare hopf.

6 10-9-25

If M is a smooth manifold, a left $G-$action is a map

$$F : G \times M \to M$$
$$(g, p) \to gp$$

such that $(g_1g_2)p = g_1(g_2p)$. A left $G-$action is free if $gp = p \Rightarrow g = e$ (stabilizer is trivial). Equivalently, the map $G \to M$ by $g \mapsto gp$ is injective.

Theorem. 7

If G is compact and $f : G \times M \to M$ is a free left G action then there is a unique smooth manifold structure on the quotient

$$M/G = M/\sim$$

$(gp \sim p)$ such that the projection $M \to M/G$ is smooth.

Example. 8

S^1 acts on $S^{2n-1} = \{z \in \mathbb{C}^n, |z| = 1\}$. by

$$t \cdot (z_1, \dots, z_n) = (e^{2\pi it}z_1, \dots, e^{2\pi it}z_n)$$

Hence $S^{2n-1}/S^1 \cong \mathbb{CP}^{n-1}$.

Example. 9

S^1 acts on $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. by

$$t \cdot (x, y, z) = ((\cos 2\pi t)x - (\sin 2\pi t)y, (\sin 2\pi t)x + (\cos 2\pi t)y, z)$$

The quotient is the closed interval $[-1, 1]$. Note the action simply identifies circles on the sphere. Note this is not a smooth manifold without boundary because the action is not free (± 1). An **orbifold** is defined like a smooth manifold, but cooord charts identify neighborhoods in 0 with neighborhoods in $\mathbb{R}^n/G$ where G is a Lie group acting linearly on R^n.

A **Lie Algebra** (over $\mathbb{R}$) is a vector space V and a bilinear pairing $[\cdot, \cdot] : V \otimes V \to V$ such that $[X, Y] = -[Y, X]$ and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Example. 10

$\mathfrak{g}l_n$ = set of n by n matrices over R. Where the Lie bracket is the commutator.

If G is a Lie group, the Lie algebra is defined as $\mathfrak{g} = T_eG$. Define the Lie bracket as the commutator of vector fields. Let M be a smooth manifold, a vector field on M is equivalent to an $\mathbb{R}$ linear map $V : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ such that it satisfies the Leibniz Rule.

Proof. If X is a vector field, and $f : M \to \mathbb{R}$ is a smooth manifold, define $Vf : M \to \mathbb{R}$ by $(Vf)(p) = (X(p))f = df_p(X(p))$. Then the Leibniz Rule follows from derivation properties. The other direction is left as exercise. ■

If V, W are two vector fields, regardeda s functions $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$. Define their commutator

$$[V,W] : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$$

by $[V,W]f = V(Wf) - W(Vf)$. Check this defines a vector field by satisfying Leibniz Rule.
To summarize, vector fields satisfy linearity and the Leibniz rule globally.
In local coordinates $x^1, \ldots, x^n$, $V = V^i \frac{\partial}{\partial x^i}, W = W^j \frac{\partial}{\partial x^j}$ where V^i are smooth functions of $x^1, \ldots, x^n$.
If f is a smooth function of $x^1, \ldots, x^n$, then

$$\begin{aligned}
[V,W]f &= V^i \frac{\partial}{\partial x^i} W^j \frac{\partial}{\partial x^j} f - W^j \frac{\partial}{\partial x^j} V^i \frac{\partial}{\partial x^i} f \\
&= V^i \frac{\partial W^j}{\partial x^i} \frac{\partial f}{\partial x^j} + V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \frac{\partial f}{\partial x^i} - W^j V^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\
\Rightarrow [V,W] &= \left(V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}
\end{aligned}$$

Let G be a Lie group, $\mathfrak{g} = T_e G$. If $g \in G$, define left multiplication $L_g : G \to G$ by $L_g(h) = gh$. This is a diffeomorphism since $L_{g^{-1}} \circ L_g = L_g \circ L_{g^{-1}} = id_G$. If $X \in \mathfrak{g}$ there is a unique left invariant vector field X^L on G characterized by $X^L(e) = X$, and if $g \in G$ then $X^L(g) = (dL_g)_e(X)$.
If $X, Y \in \mathfrak{g}$, define

$$[X,Y] = [X^L, Y^L]$$

Note if we instead defined it using X^R, Y^R, we'd get the same thing: $[X^R, Y^R] = [X^L, Y^L]$

Example. 11

$$G = GL(n,\mathbb{R}) \subset_{open} \{n \times n \text{ matrices}\}$$

Claim: If A, B are matrices, then

$$[A,B] = AB - BA$$

Proof. Let $\{x^i_j\}$ be coordinates on the space of $n \times n$ matrices. Matrix multiplication is $(XY)^i_j = X^i_k Y^k_j$.

$$A^L(X) = X^i_k A^k_j \frac{\partial}{\partial x^i_j}$$

$$B^L(X) = X^p_r B^r_q \frac{\partial}{\partial x^p_q}$$

Thus,

$$\begin{aligned}
[A^L, B^L] &= X^i_k A^k_j \frac{\partial}{\partial x^i_j} (X^p_r B^r_q) \frac{\partial}{\partial x^p_q} - X^p_r B^r_q \frac{\partial}{\partial x^p_q} (X^i_k A^k_j) \frac{\partial}{\partial x^i_j} \\
&= \cdots = (AB)^L - (BA)^L
\end{aligned}$$

■

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Lie algebra $\mathfrak{g} = T_eG$. Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ define for $X, Y \in \mathfrak{g}$: $[X,Y] = [X^L, Y^L](e)$. $X^L(e) = X, Y^L(e) = Y$ are left invariant vector fields. One checks $[X^L, Y^L]$ is left invariant. Another way to write the definition of Lie bracket is

$$[X,Y]^L = [X^L, Y^L]$$

and $X^L(g) = (dL_g)_e(X)$.
Fact: for any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there exists Lie group G whose Lie algebra is $\mathfrak{g}$. Note: Non-isomorphic Lie groups can have isomorphic Lie algebras.

Example. 12
$SU(2)$ is the set of 2 by 2 complex matrices A with $A^* = -A, \mathrm{Tr}\ A = 0$. Real basis:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Call them (a, b, c). One checks

$$[a,b] = 2c, [b,c] = 2a, [c,a] = 2b$$

Now consider $SO(3)$. Write a basis

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Call them (x, y, z). One checks $[x,y] = z, [y,z] = x, [z,x] = y$. Their Lie algebras are isomorphic. But $SU(2), SO(3)$ are not isomorphic (i.e. no diffeomorphism which is a group isomorphism). They are not even homeomorhpic as they have different fundamental groups.

Let M, N be smooth manifolds, and X be a vector field on M. Let $\phi : M \to N$ be a diffeomorphism. Define vector field ϕ_*X on N, the push forward of X as follows: if $q \in N$ let $p \in M$ be the point with $\phi(p) = q$, then $(\phi_*X)(q) = d\phi_p(X(p))$. Vector fields X, Y are $f-related$ if for all $p \in M$,

$$df_p(X(p)) = Y(f(p))$$

Given X, an $f-related$ vector field Y might not exist.
1. If f is not injective, if $q \in N$ and $|f^{-1}(q)| > 1$ might not be able to define $Y(q)$ consistently.
2. Even if one can define Y consistently on $f(M)$, might not be able to extend smoothly over the rest of N. (OK if M is compact).

Example. 13
Let G, H be Lie groups and let $f : G \to H$ be a Lie group homomorphism. Let $X \in \mathfrak{g}$, let $Y = df_e(X) \in \mathfrak{h}$. Then X^L, Y^L are $f-$related.

Proof. Let $g \in G$. Check

$$df_g(X^L(g)) = Y^L(f(g))$$

We have $X^L(g) = d(L_g)_e(X), Y^L(f(g)) = (dL_{f(g)})_e(df_e(X))$. Our equation becomes

$$df_g(d(L_g)_e(X)) = dL_{f(g)}(df_e(X)) \Leftrightarrow d(f \circ L_g)_e(X) = d(L_{f(g)} \circ f)_e(X)$$

i.e .$f \circ L_g = L_{f(g)} \circ f$, which is clearly true. ■

Lemma: $f : M \to N$ smooth, X, X', Y, Y' vector fields on M. If X, Y are $f-$related and X', Y' are $f-$related then $[X, X']$ and $[Y, Y']$ are $f-$related.
Corollary: If $f : G \to H$ is a Lie group homomorphism then $df_e : \mathfrak{g} \to \mathfrak{h}$ respects the Lie bracket, so is a morphism of Lie algebas.
A representation of G is homomorphism $G \to GL_n(\mathbb{C})$, and a representation of $\mathfrak{g}$ is a morphism of algebras $\mathfrak{g} \to \mathfrak{gl_n}(\mathbb{C})$.
If $\pi_1(G) = 0$ and G connected, then there is a bijection between representations of G and representations of $\mathfrak{g}$.

7.1 Flow of a vector field

M smooth manifold, X a vector field on M. Then a trajectory of X is a smooth map $\gamma : (a, b) \to M$ such that

$$\gamma'(t) = X(\gamma(t))$$

for all $t \in (a, b)$.

Theorem. 8

M smooth manifold, X vector field over M, $p \in M$. Then there exists neighborhood U of p in M and $\epsilon > 0$ and a smooth map $\Phi : (-\epsilon, \epsilon) \times U \to M$ such that $\Phi(0, q) = q$ and for fixed $q \in U$

$$\frac{d}{dt}\Phi(t, q) = X(\Phi(t, q))$$

Given U, ϵ, Φ is unique and exists. Intuitively, this extends the idea of finding $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p, d/dt(\gamma(t)) = X(\gamma(t))$ to a neighborhood on the manifold.
Note: might not be able to take $\epsilon = \infty$, consider $y' = y^2$.
If M compact, we can take $U = M, \epsilon = \infty$.

Proof. Reduce (U_p) to finite subcover U_{p_i}. Let $\epsilon = \min \epsilon_{p_i}$, fundamental theorem gives a unique map $\Phi : (-\epsilon, \epsilon) \times M \to M$ such that the condition holds. Extend to $\Phi : \mathbb{R} \times M \to M$ by iteration: if $t \in \mathbb{R}, p \in M$ define $\Phi(t, p) = \Phi(\epsilon/2, \Phi(t - \epsilon/2, p))$. ■

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M manifold, X vector field, there exists unique

$$\Phi : \mathbb{R} \times M \to \mathbb{R}$$

such that $\Phi(0, \cdot) = Id_M$ and for fixed $p \in M$ we have

$$\frac{d}{dt}\Phi(t,p) = X(\Phi(t,p))$$

If M is not compact, Φ is locally defined. For $t \in \mathbb{R}$, $\phi_t : M \to M$ given by $p \mapsto \Phi(t,p)$, so we have $\frac{d}{dt}\phi_t(p) = X(\phi_t(p))$.
Observe if $s, t \in \mathbb{R}$ then

$$\phi_{s+t} = \phi_t \circ \phi_s$$

Proof. For fixed $p \in M$, we have two maps $\mathbb{R} \to M$ given by

$$\gamma_1(t) = \phi_{s+t}(p)$$
$$\gamma_2(t) = (\phi_t \circ \phi_s)(p)$$

■

Want to show $\gamma_1 = \gamma_2$ True for $t = 0$.

$$\frac{d}{dt}\gamma_1(t) = \frac{d}{dt}\phi_{s+t}(p) = X(\phi_{s+t}(p)) = X(\gamma_1(t))$$
$$\frac{d}{dt}\gamma_2(t) = \frac{d}{dt}(\phi_t \circ \phi_s(p)) = X(\phi_t(\psi_s(p)) = X(\gamma_2(t))$$

Since $\gamma_{1,2}$ satisfy the same ode, they are equal.
Corollary: $\phi_t : M \to M$ is a diffeomorphism: just take ϕ_{-t}.

Theorem. 9
Let G be a Lie group (not necessarily cmopact). Let $X \in \mathfrak{g}$. Ket X^L be the left invariant vector field on G extending X. Claim: The flow of X^L is defined for all time.

Proof. There exists $\epsilon > 0$ and a map

$$\gamma_e : (-\epsilon, \epsilon) \to G$$

with $\gamma_e(0) = e$, $\frac{d}{dt}(\gamma_e(t)) = X^L(\gamma_e(t))$. For any $g \in G$ we find a map

$$\gamma_g : (-\epsilon, \epsilon) \to G$$

for the same ϵ. with $\gamma_g(0) = g$ and $\frac{d}{dt}\gamma_g(t) = X^L(\gamma_g(t))$. To do this, define

$$\gamma_g(t) = g \cdot \gamma_e(t)$$

Check that this works:

$$\gamma_g(0) = g \cdot \gamma_e(0) = g \cdot e = g$$

and

$$\begin{aligned}
\frac{d}{dt}\gamma_g(t) &= \frac{d}{dt}(L_g(\gamma_e(t))) \\
&= (L_g)_*(\frac{d}{dt}(\gamma_e(t))) \\
&= (L_g)_* X^L(\gamma_e(t)) \\
&= X^L(L_g(\gamma_e(t))) = X^L(\gamma_g(t))
\end{aligned}$$

So we have a map

$$\Phi : (-\epsilon, \epsilon) \times G \to G$$

with $\Phi(0, \cdot) = id_G$ and $\frac{d}{dt}\Phi(t, g) = X^L(\Phi(t, g))$. By fundamental theorem of ODEs Φ is smooth. By iterating, get $\Phi : \mathbb{R} \times G \to G$ satisfying the above.
Define $\gamma_X : \mathbb{R} \to G$ to be the path satisfying

$$\gamma_X(0) = e, \frac{d}{dt}(\gamma_X(t)) = X^L(\gamma(t))$$

This is a group homomorphism.

$$\gamma_X(s + t) = \gamma_X(s)\gamma_X(t)$$

γ_X is the one parameter subgroup. Warning: it might not be an embedding. ■

We get the same γ_X if we used X^R.

Example. 14

Exponential Map. $\exp : \mathfrak{g} \to G$ defined by $\exp(X) = \gamma_X(1)$. If $G = GL_n(\mathbb{R})$ and $A \in \mathfrak{g} = M_n(\mathbb{R})$ then

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Warning: exp might not be surjective. If $G = SL_2(\mathbb{R})$

$$g = \begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix} \in G$$

There does not exist A with $e^A = g$.
Define the **Lie derivative**.
Let V be a vector field on M. Let $\{\phi_t\}$ denote the flow of V. ($\phi_t(p)$ defined locally for each p). Let W be another vector field on M, define the Lie derivative of W wrt V to be the vector field

$$\mathcal{L}_V W = \frac{d}{dt}\Big|_{t=0} \phi_t^* W$$

where $\phi_t^* = (\phi_{-t})_*$.

Theorem. 10

$\mathcal{L}_V W = [V, W]$.

Proof. Suppose $V(p) \neq 0$. Then we can take a neighborhood of p in which $V(p) \neq 0$. Choose a coord chart in this neighborhood in which $V(p) = \frac{\partial}{\partial x^1}$. Claim: we can make a new coord chart containing p such that $V = \frac{\partial}{\partial x^1}$ on the entire chart.
Old chart: $\phi : U \to \hat{U} \subset \mathbb{R}^n$ with $p \mapsto 0$. $\psi^{-1} : (nbhd\, in\, \mathbb{R}^n) \to M$ by $\psi^{-1}(x^1, \ldots, x^n) =$ flow of V starting $\phi(0, x^2, \ldots, x^n)$ for time x^1. ψ is a new coord chart in which $V = \frac{\partial}{\partial x^1}$ by construction.

$$\begin{aligned} V &= \frac{\partial}{\partial x^1} \\ W &= W^i \frac{\partial}{\partial x^i} \\ [V, W] &= \frac{\partial}{\partial x^i}(W^i \frac{\partial}{\partial x^i}) + \frac{\partial W^i}{\partial x^1}\frac{\partial}{\partial x^i} = \frac{\partial W^i}{\partial x^1}\frac{\partial}{\partial x^i} \end{aligned}$$

$(\phi_t)_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$.

$$(\phi_t^* W)(0) = W(t,0,\dots,0) = W^i(t,0,\dots,0)\frac{\partial}{\partial x^i}$$

$$(\frac{d}{dt}\phi_t^* W)(0) = \frac{d}{dt}\Big|_{t=0} W^i(t,0,\dots,0)\frac{\partial}{\partial x^i} = \frac{\partial W^i}{\partial x^1}(0)\frac{\partial}{\partial x^i}$$

■

Theorem. 11

M compact manifold. Then $[V,W]=0 \Leftrightarrow$ the flows of V and W commute. i.e. if $\{\phi_t\}$ denotes the flow of V and $\{\psi_s\}$ denotese the flow of W then $\psi_s \circ \phi_t = \phi_t \circ \psi_s$.

Proof. $(\Leftarrow)$ Suppose that the flows commute. Let $f : M \to \mathbb{R}$. Need to show $[V,W]f = 0$ i.e. $V(Wf) = W(Vf)$. Fix $p \in M$.

$$\begin{aligned}
(Vf)(p) &= \frac{d}{dt}\Big|_{t=0}(f(\phi_t(p))\\
(W(V(f))(p) &= \frac{d}{ds}\Big|_{s=0}(\frac{\partial}{\partial t}\Big|_{t=0} f(\phi_t(\psi_s(p))))\\
&= \frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\phi_t(\psi_s(p))) = \frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\psi_s(\phi_t(p)))
\end{aligned}$$

$(\Rightarrow)$ Suppose $[V,W]=0$, show that the flows commute. Enough to show that $\psi_s \circ \phi_t = \phi_t \circ \psi_s$ for $|s|,|t|$ small. Consider a nhbd of $p \in M$. Suppose $V(p) \neq 0$. Choose coords in which $V = \frac{\partial}{\partial x^1}$ so $\phi_t(x^1,\dots,x^n) = (x^1+t,\dots,x^n)$. Write $W = W^i \frac{\partial}{\partial x^i}$. Then

$$[V,W] = \frac{\partial W^i}{\partial x^1}\frac{\partial}{\partial x^i} = 0$$

which implies $W^1,\dots,W^n$ do not depend on x^1. ■

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Let M be a manifold of dimension n. A (rank k) vector bundle is a smooth manifold E of dimension $n+k$ and a smooth map $\pi : E \to M$ such that for each $p \in M$, the fiber $\pi^{-1}(p)$ has the structure of a real k-dimensional vector space, and
(local triviality) For each $p \in M$, there is a neighborhood $p \in U$ and a diffeomorphism

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[d, "\pi_1"] \\ U \arrow[r, "id"] & U \end{tikzcd}$$

such that for each $p \in U$, the resriction

$$\phi_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^k$$

is an isomorphism of vector spaces.

Example. 15
the trivial vector bundle is $E = M \times \mathbb{R}^k$ and $\pi = \pi_1 : M \times \mathbb{R}^k \to M$.

Example. 16
The tangent bundle TM is a rank n tangent bundle over M. Let's check local triviality. Let $p \in M$, let $\psi : U \to \hat{U} \subset \mathbb{R}^n$ be a coord chart. Write coord functions as $x^1, \ldots, x^n$. If $q \in U$ and $V = T_q M$, write $V = V^i \partial_{x^i}$. Define $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by $\phi(q, V) = (q, (V^1, \ldots, V^n))$.

Let E_1, E_2 be vector bundles on M with maps π_1, π_2. A vector bundle isomorphism $E_1 \to E_2$ is a diffeomorphism such that for each $p \in M$, $\phi_{\pi^{-1}(p)} : \pi^{-1}(p) \to \pi_2^{-1}(p)$ is a vector space isomorphism. A vector bundle $\pi : E \to M$ is trivial if there is a vector bundle isomorphism $E \cong M \times \mathbb{R}^k$. When is a vector bundle trivial?
A **section** of $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$. That is, for each $p \in M$, $s(p) \in E_p$. For example, a vector field is a section of TM.

Remark. $\pi : E \to M$ is trivial iff there exists sections $s_1, \ldots, s_k$ such that for each $p \in M$, $s_1(p), \ldots, s_k(p) \in E_p$ are linearly independent. Proof: For forward, let $\phi : E \to M \times \mathbb{R}^k$. Define $s_i(p) = \phi^{-1}(p, e_i)$ where $e_1, \ldots, e_k$ are a basis for $\mathbb{R}^k$. Reverse direction define $\phi : E \to M \times \mathbb{R}^k$ by $\phi(\xi) = (p, q_1, \ldots, q_k)$ where $\xi \in E_p$, $\xi(p) = a_1 s_1(p) + \cdots + a_k s_k(p)$.

Corollary. 1
If $\dim M > 0$ and $\chi(M) \neq 0$ then TM is not trivial.

Proof. If $\chi(M) \neq 0$, then every section of TM has a zero by poincare hopf. Since we can't find a nonvanishing vector field, we certainly can't find $n = \dim M$ vector fields, that are linearly independent at each point. ■

Definition. If $E \to M$ is a vector bundle, an **orientation** of p is an orientation of E_p for each $p \in M$ which depends continuously on p. Continuity means that if $s_1, \ldots, s_k$ are sections over U with $s_1(p), \ldots, s_k(p)$ linearly independent for each $p \in U$, then either $s_1(p), \ldots, s_k(p)$ are an oriented basis for E_p for each $p \in U$, or $s_1(p), \ldots, s_k(p)$ are not an oriented basis for all $p \in M$.
For example, M is orientable iff TM as a vector bundle has an orientation. If E is trivial then E is orientable since it has orientation induced from M.

Example. 17

Mobius band gives a rank 1 vector bundle over S^1. Let $M = \mathbb{R} \times \mathbb{R}/$,(x+1, y) $\sim (x, -y)$. Define $\pi : M \to S^1$ by $[(x, y)] \mapsto [x]$.

9.1 Operations on vector bundles

Let $\pi_1 : E_1 \to M$ be of rank k_1, and $\pi_2 : E_2 \to M$ be of rank k_2. The **Whitney** (direct) sum $E_1 \oplus E_2 \to M$ is of rank $k_1 + k_2$ defined as

$$E_1 \oplus E_2 = \{(p, V_1, V_2) \mid p \in M, V_1 \in (E_1)_p, V_2 \in (E_2)_p\}$$

Vector space structure on $(E_1 \oplus E_2)_p = (E_1)_p + (E_2)_p$ is the direct sum.
One can also define the dual as

$$E^* = \{(p, \alpha) \mid p \in M, \alpha \in E_p^*\}$$

For example the cotangent $T^*M = (TM)^*$.
One can define the tensor product as $E_1 \otimes E_2$ with rank $k_1 k_2$ on each fiber

$$(E_1 \otimes E_2)_p = (E_1)_p \otimes (E_2)_p$$

What is a tensor product again?
Let V, W be such that $\dim V = m, \dim W = n$. Then $V \otimes W$ is a vector space with bilinear map $\otimes : V \times W \to V \otimes W$.
Defintion 1: $V \otimes W$ is the set of all finite linear combinations $\sum_{i=1}^k v_i \otimes w_i$ modulo relations

$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

Definition. Let M be a smoorth manifold. A 0-form on M is a smooth function $f : M \to \mathbb{R}$. A 1-form on M is a section α of T^*M.

$d : \{0 - forms\} \to \{1 - forms\}$ takes $f : M \to \mathbb{R}$ to $\alpha : p \mapsto df_p \in T_p^*M$. In local coords $x^1, \dots, x^n$, $dx^1, \dots, dx^n$ are a basis for the cotangent space at each point, $df = \frac{\partial f}{\partial x^i} dx^i$. Check $V = V^j \frac{\partial}{\partial x^j}$

$$df(v) = v^j \frac{\partial f}{\partial x^i} \in \mathbb{R}$$

We can integrate a 1-form over a parametrized curve. Let α be a 1-form and let $\gamma : [a, b] \to M$.

Definition.

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma'(t))dt$$

Lemma. 1

This is unchanged under reparametrization fixing the endpoints.

More precisely, if $\phi : [c, d] \to [a, b]$ is a diffeomorphism with $\phi(c) = a, \phi(d) = b$ then

$$\int_{\gamma \circ \phi} \alpha = \int_\gamma \alpha$$

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Theorem. 12

If $\varphi : [a,b] \to [c,d]$ is a diffeomorphism with $c < d$. Then

$$\int_{\gamma \circ \varphi} \alpha = \int_{\gamma} \alpha$$

Proof.

$$\begin{aligned}
\int_{\gamma\circ\varphi} \alpha &= \int_c^d \alpha((\gamma\circ\varphi)'(t))dt \\
&= \int_c^d \alpha(\gamma'(\varphi(t))\varphi'(t))dt \\
&= \int_c^d \alpha(\gamma'(\varphi(t))\varphi'(t)dt \\
&= \int_a^b \alpha(\gamma'(u))du \\
&= \int \gamma\alpha
\end{aligned}$$

■

A 0-form on M is a smooth function $f : M \to \mathbb{R}$. Define

$$d : 0 - forms \to 1 - forms$$

$$df(p)(v) = vf$$

for $v \in T_pM$.

Theorem. 13

The fundamental theorem of line integrals. If $f : M \to \mathbb{R}$, and $\gamma : [a,b] \to M$, then

$$\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a))$$

Proof.

$$\begin{aligned}
\int_{\gamma} df &= \int_a^b df(\gamma'(t))dt \\
&= \int_a^b \gamma'(t)fdt \\
&= \int_a^b d\gamma(\frac{d}{dt})fdt \\
&= \int_a^b (f\circ\gamma)'(t)dt \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

■

Definition. A 1-form α is exact if $\alpha = df$ for some $f : M\mathbb{R}$. Proposition: if α is exact, $\int_\gamma \alpha$ depends only on $\gamma(b), \gamma(a)$.

Proof. WLOG M is connected. Pick $p_0 \in M$. Define $f(p_0) = 0$. Let $\gamma : [a, b] \to M$ be a path with $\gamma(a) = p_0, \gamma(b) = p$. Define

$$f(p) = \int_\gamma \alpha$$

Then f is well defined since it only depends on the end points of γ. Need to check that $\alpha = df$. Fix $p \in M$. Fix $\gamma : p_0 \mapsto p$. If $q \in U$, let ψ be a path in U from p to q. Then $f(q) = f(p) + \int_\psi \alpha$. In local coords, $\alpha = \alpha_i dx^i$. Let's calculate

$$\frac{\partial}{\partial x^1} f(q)\big|_{(0,\ldots,0)} = \frac{d}{ds}\big|_{s=0} f(s, 0, \ldots, 0)$$

For $s > 0$ use path $\phi_s : [0, s] \to M$, $\psi_s(t) = (t, 0, \ldots, 0)$. $f(s, 0, \ldots, 0) = \int_{\psi_s} \alpha = \int_0^s \alpha_1(t, 0, \ldots, 0)dt$

$$= \frac{d}{ds}\big|_{s=0} \int_0^s \alpha_1(t, 0, \ldots, 0)dt$$

■

Proposition. 1

Let $\alpha = \alpha_i dx^i$ be a 1-form on $\mathbb{R}^n$. Then α is exact iff

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}$$

Proof. Suppose $\alpha = df$. Then $\alpha_i = \frac{\partial f}{\partial x^i}$. So

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial \alpha_j}{\partial x^i}$$

Suppose $\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}$. Then define

$$f(x^1, \ldots, x^n) = \int_0^{x^1} \alpha_1(t, 0, \ldots, 0)dt + \int_0^{x^1} \alpha_n(x^1, t, 0, \ldots, 0)dt + \cdots + \int_0^{x^n} \alpha_n(x^1, \ldots, x^{n-1}, t)dt$$

Need to check that $\frac{\partial f}{\partial x^i} = \alpha_i$. To simplify notation let $n = 2$.

$$f(x^1, x^2) = \int_0^{x^1} \alpha_1(t, 0)dt + \int_0^{x_2} \alpha_2(x^1, t)dt$$

and

$$\frac{\partial f}{\partial x^2} = \frac{\partial}{\partial x^2} \int_0^{x_2} \alpha_2(x^1, t)dt = \alpha_2(x^1, x^2)$$

and

$$\begin{aligned}
\frac{\partial f}{\partial x^1} &= \alpha_1(x^1, 0) + \int_0^{x_2} \frac{\partial \alpha_2}{\partial x^1}(x^1, t)dt \\
&= \alpha_1(x^1, 0) + \int_0^{x_2} \frac{\partial \alpha_1}{\partial x^2}(x^1, t)dt \\
&= \alpha_1(x^1, 0) + \alpha_1(x^1, x^2) - \alpha_1(x^1, 0) = \alpha_1(x^1, x^2)
\end{aligned}$$

■

Example. 18

This does not generalize to manifolds other than $\mathbb{R}^n$. On $\mathbb{R}^2\backslash\{0\}$, consider

$$\alpha = \frac{xdy - ydx}{x^2+y^2}$$

so $\alpha_1 = \frac{-y}{x^2+y^2}$, $\alpha_2 = \frac{x}{x^2+y^2}$. Then

$$\frac{\partial \alpha_1}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$
$$\frac{\partial \alpha_2}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

But α is not exact. Consider $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ by

$$\gamma(t) = (\cos t, \sin t)$$

Then

$$\int_\gamma \alpha = \int_0^{2\pi} \frac{xdy - ydx}{x^2+y^2}(-\sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y})dt = \int_0^{2\pi}(\cos^2 t + \sin^2 t)dt = 2\pi$$

Should be 0.

Remark. If $n > 2$ and $\alpha = \alpha_i dx^i$ is a 1-form on $\mathbb{R}^n - \{0\}$ satisfying $\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}$, then α is exact. (proved using de Rham)

Definition. A 2-form on M is a smooth section β of $(TM \otimes TM)^*$ which is antisymmetric. That is, for each $p \in M$, $\beta(p) : T_pM \times T_pM \to \mathbb{R}$ is a function such that $\beta(p)$ is bilinear. (so $\beta(p)$ descends to $T_pM \otimes T_pM$).

$$\beta(p)(v_1 + v_2, w) = \beta(p)(v_1, w) + \beta(p)(v_2, w)$$
$$\beta(p)(cv, w) = c\beta(p)(v, w)$$

Antisymmetric:

$$\beta(v, w) = -\beta(p)(w, v)$$

Claim: all two forms have the form $\sum_{i<j} \beta_{ij} dx^i \wedge dx^j$.